B.Sc. (Semester - 5)
Subject: Physics
Classical Mechanics
Course: US05CPHY21

UNIT-III Variational Principle

Configuration Space:

In the case of motion of a single particle we can represent its trajectory in the three dimensional space by specifying its variable. For a system of N particles described by $3N$ space coordinates with k equations of constraint in the real space. It is difficult to visualize the motion of the entire system. It is therefore convenient to describe the state of a system having $3N - k = n$ coordinates in a hypothetical $n -$ dimensional space. This is an extension of the three dimensional to the $n -$ dimensional geometry. The state of the system is then described by a point having generalised coordinates q_j, where $i = 1,2, \ldots \ldots n$. The point is called the *system point* and the $n -$ dimensional space is known as the *configuration space*.

At some instant, the state of the system changes and it will be represented by some other point in the configuration space. Thus, the system point moves in the configuration space tracing out a curve. This curve represents the path of motion of the entire system. The motion of the system means the motion of the system point along this path in the configuration space.

Some Techniques of Calculus of Variation:

The basic problem of calculus of variation is to find a path $y = y(x)$ in one dimension between x_1 and x_2, such that the line integral of some function $f(y, y', x)$. Where $y' = \frac{dy}{dx}$ is an extremum. i.e. maximum or minimum.

Statement: *For a function $f(y, y', x)$, the line integral*

$$J = \int_{x_1}^{x_2} f(y, y', x)\, dx \qquad \ldots (3.1)$$

along the path $y = y(x)$ between x_1 and x_2 is to be extremum.

Let (x_1, y_1) and (x_2, y_2) be two points in the space as shown in fig.3.1. There are two varied paths between two extreme points $y(x_1) = y_1$ and $y(x_2) = y_2$.

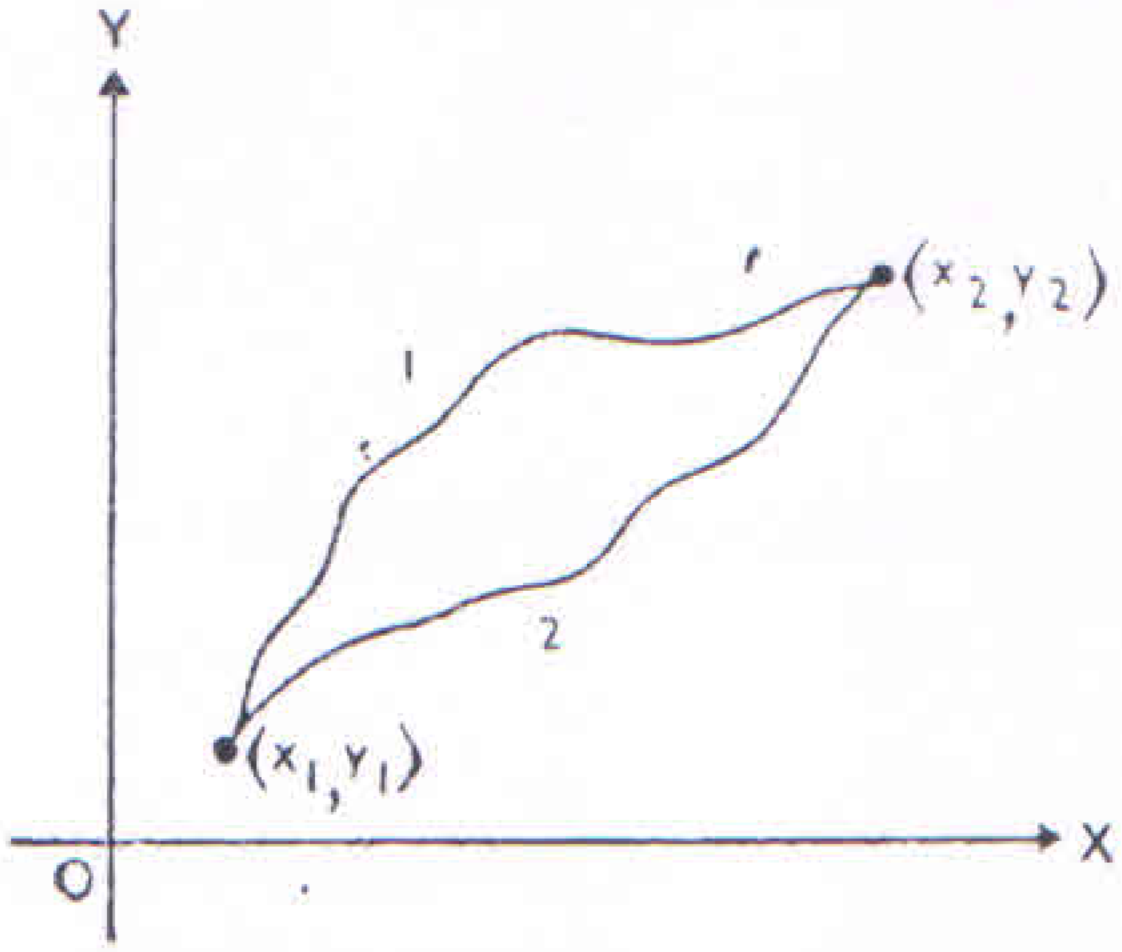

Fig: 3.1

In order to find a path give an extremum value, we associate a parameter α with all the possible path. Then y will be a function of both the independent variable x and the parameter α.

$$\therefore y(\alpha, x) = y(0, \alpha) + \alpha\, \eta(x) \qquad \text{...(3.2)}$$

Here, $\eta(x) = \frac{\partial y}{\partial \alpha}$ and $\eta(x)$ is some function of x for which the function itself vanishes at both $x = x_1$ and $x = x_2$.

$$\therefore\ \eta(x_1) = \eta(x_1) = 0 \text{ at the extremum.}$$

Now using equation (3.2) in (3.2), we get

$$J(\alpha) = \int_{x_1}^{x_2} f[y(\alpha, x), y'(\alpha, x), x]\, dx \qquad \text{...(3.3)}$$

Then, the condition that $J(\alpha)$ has an extremum value is

$$\left[\frac{\partial J}{\partial \alpha}\right]_{\alpha=0} = 0 \qquad \text{...(3.4)}$$

Differentiate equation (3.3) with respect to α , we get

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\int_{x_1}^{x_2} f(y, y', x)\, dx\right]$$

$$\therefore\ \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha}\right] dx$$

The end points are fixed. There is no variation at the end points. Hence, $\frac{\partial x}{\partial \alpha} = 0$

$$\therefore\ \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right] dx$$

$$\therefore\ \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial \alpha\, \partial x}\right] dx \qquad \text{...(3.5)}$$

Integrating the second term in the integrand by parts, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right) dx = \frac{\partial f}{\partial y'}\frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha} dx \qquad ...(3.6)$$

But,

$$\frac{\partial J}{\partial \alpha} = \eta(x), Hence\ \frac{\partial y}{\partial \alpha}\bigg|_{x_1}^{x_2} = \eta(x_2) - \eta(x_1) = 0 \qquad ...(3.7)$$

Thus, the first term on the R.H.S of equation (3.6) vanishes. Hence, Equation (3.5) becomes

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\frac{\partial y}{\partial \alpha}\right] dx$$

$$\therefore \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \eta(x)\, dx \qquad ...(3.8)$$

Since $\eta(x)$ is an arbitrary function such that $\eta(x_1) = \eta(x_1) = 0$. The integral of equation (3.8) must vanish for $\alpha = 0$. Thus, we have

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad ...(3.9)$$

This equation is called Euler's equation and it represents the necessary condition that the integral J has the extremum value.

Euler's equation can be generalised when f is a function of several dependent variables.

$$\therefore f = f[y_i(x), y_i'(x), x], \quad i = 1,2, \dots \dots n \qquad ...(3.10)$$

Here, $y_i(\alpha, x) = y_i(0, x) + \alpha\, \eta_i(x)$ ...(3.11)

Hence, Euler's equation becomes,

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \qquad ...(3.12)$$

More generally,

$$f = f[y_i(x_j), y_i'(x_j), x_j]$$

Where, $i = 1,2, \dots \dots \dots n$ and $j = 1,2, \dots \dots \dots k$.

The Euler's Lagrange's equation takes the form

$$\frac{\partial f}{\partial y_i} - \sum_{j=1}^{k} \frac{\partial}{\partial x_j}\frac{\partial f}{(\partial y_i/\partial x_j)} = 0 \qquad ...(3.13)$$

Euler's equation (3.9) can also be put into another form. Let us consider

$$\frac{d}{dx}f(y, y', x) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial x}$$

$$\therefore \frac{df}{dx} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} \qquad ...(3.14)$$

Now,

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

Substituting the value of $y''\frac{\partial f}{\partial y'}$ from equation (3.14) in above equation , we get

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)=\frac{df}{dx}-\frac{\partial f}{\partial x}-y'\frac{\partial f}{\partial y}+y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$\therefore\ \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)=\frac{df}{dx}-\frac{\partial f}{\partial x}-y'\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}\right] \qquad ...(3.15)$$

Using equation (3.9) in (3.15), we get

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=0 \qquad ...(3.16)$$

This is called the second form of Euler's equation. When f does not depend upon x then,

$$f-y'\frac{\partial f}{\partial y'}=const. \qquad ...(3.17)$$

The δ –Notation:

The results of the calculus of variation are expressed in terms of δ- notation as follows:

We have

$$\frac{\partial J}{\partial \alpha}=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}\,dx$$

Multiplying both the sides by $d\alpha$, we get

$$\frac{\partial J}{\partial \alpha}d\alpha=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\frac{\partial y}{\partial \alpha}d\alpha\,dx \qquad ...(3.18)$$

$$\text{Taking}\frac{\partial J}{\partial \alpha}d\alpha=\delta J,\quad and\quad \frac{\partial y}{\partial \alpha}d\alpha=\delta y \qquad ...(3.19)$$

$$\therefore\ \delta J=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\delta y\,dx \qquad ...(3.20)$$

The condition of extremum becomes

$$\delta J=\delta\int_{x_1}^{x_2}f(y,y',x)\,dx=0 \qquad ...(3.21)$$

Taking δ inside the integral, we get

$$\delta J=\int_{x_1}^{x_2}\delta f\,dx=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial y'}\delta y'\right]dx \qquad ...(3.22)$$

Now,

$$\delta y'=\delta\left(\frac{dy}{dx}\right)=\frac{d}{dx}(\delta y)$$

Hence,

$$\delta J=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\delta y+\frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\right]dx \qquad ...(3.23)$$

Integrating second term by parts, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\,dx = \frac{\partial f}{\partial y'}\delta y\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\ dx$$

But, $\delta y|_{x_1}^{x_2} = 0$

$$\therefore \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\frac{d}{dx}(\delta y)\,dx = -\int_{x_1}^{x_2} \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\delta y\ dx \qquad \dots(3.24)$$

Substituting this equation (3.24) in equation (3.23), we get

$$\delta J = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] \delta y\ d\alpha \qquad \dots(3.25)$$

Using the condition of variation, we get

$$\delta J = 0$$

This gives the Euler's equation in δ - notation.

Applications of The Variational Principle:

1. **To show that the shortest distance between two points in a plane is a straight line.**

Consider the XY –plane. The length of the element is given by

$$ds^2 = dx^2 + dy^2$$

$$\therefore ds = \sqrt{dx^2 + dy^2}$$

$$\therefore ds = dx\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \qquad \dots(3.26)$$

The coordinates at the ends points are (x_1, y_1) and (x_2, y_2)

$$\therefore J = \int_1^2 ds = \int_{x_1}^{x_2} \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \qquad \dots(3.27)$$

The condition for shortest path is that the integral J should be minimum.

Comparing this equation with variational principle, we obtain

$$f = \sqrt{1 + y'^2} \qquad \dots(3.28)$$

Where, $y' = \frac{dy}{dx}$, Now

$$\frac{\partial f}{\partial y} = 0 \quad and \quad \frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1 + y'^2}}2y' = \frac{y'}{\sqrt{1 + y'^2}}$$

The Euler's equation is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\therefore \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0$$

$$\therefore \frac{y'}{\sqrt{1+y'^2}} = const.C$$

$$\therefore y'^2 = C^2(1+y'^2) = C^2 + C^2y'^2$$

$$\therefore y'^2(1-C^2) = C^2$$

$$\therefore y' = \frac{C}{\sqrt{1-C^2}} = a\,, const$$

Integrating above equation, we get

$$y = a\,x + b \qquad \text{... (3.29)}$$

This is the equation of straight line.

2. **The Brachistochrone OR shortest time problem:**

We consider a particle which moves in a constant conservative force-field $\vec{F}$ as shown in Fig.(3.2). Suppose that the particle is initially at rest at some point and moves to some other point (x_1, y_1) under the action of the force.

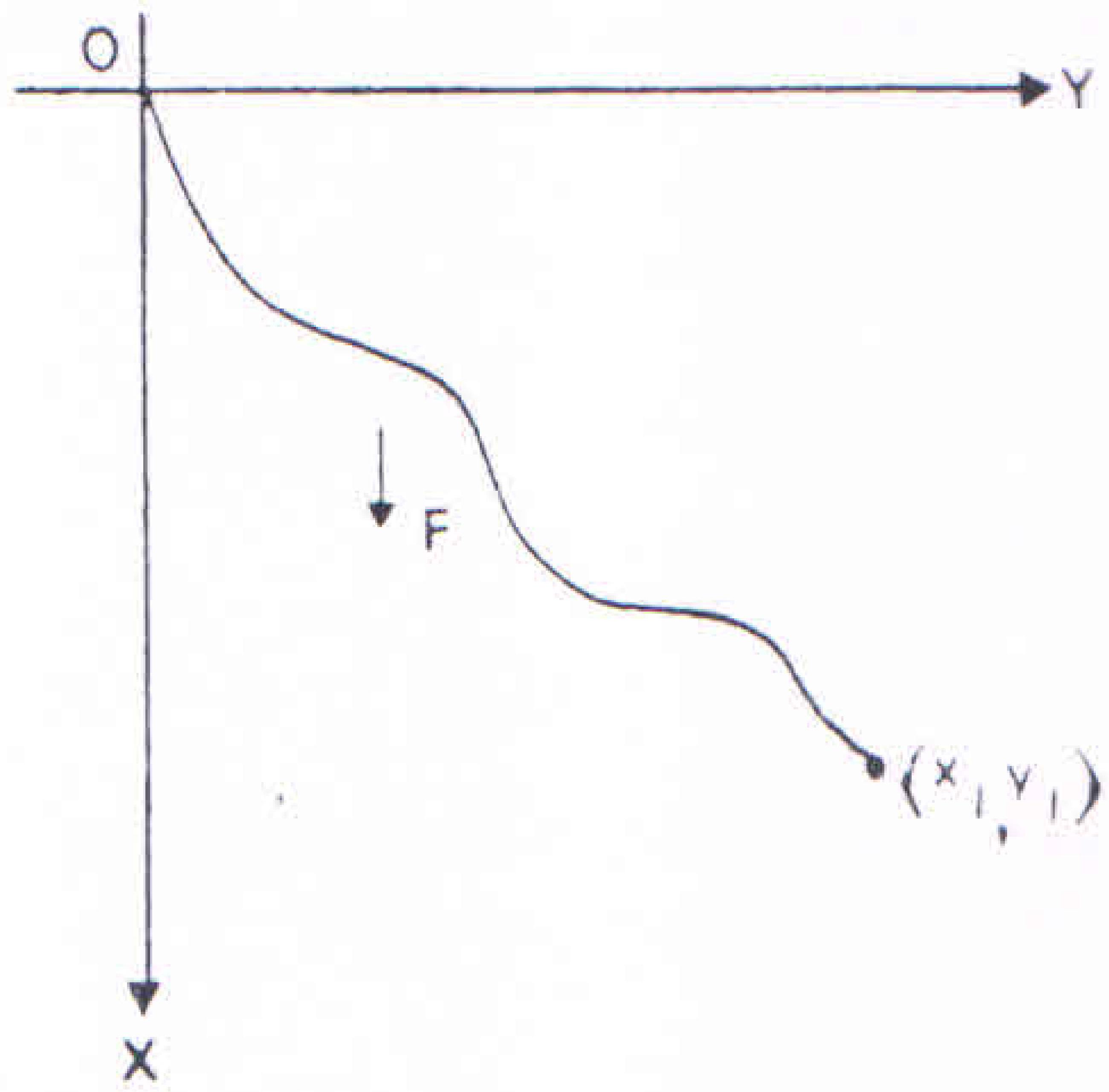

Fig: 3.2

Let v be the speed of the particle along the curve, the time of travelling is given by

$$t_{12} = \int_1^2 \frac{ds}{v} \qquad \text{... (3.30)}$$

If the frictional force is ignored, the total energy of the particle is

$$T + V = const.$$

Now,

$$T = \frac{1}{2}mv^2 \quad and \;\; V = -mgx$$

From the conservation principle for the energy of the particle we find

$$\frac{1}{2}mv^2 \quad = mgx$$

$$\therefore v = \sqrt{2gx} \qquad \text{... (3.31)}$$

Using equation (3.31) in (3.30), we have

$$t_{12} = \int_1^2 \frac{ds}{v} = \int_0^{x_1} \sqrt{\frac{(1+y'^2)}{2gx}}\, dx \qquad \text{... (3.32)}$$

$$\therefore\ t_{12} = \int_0^{x_1} f\ dx$$

Where,

$$f = \sqrt{\frac{(1+y'^2)}{x}} \qquad \text{... (3.33)}$$

Factor $(2g)^{-1/2}$ does not affect the final equation.

The Euler's-Lagrange's equation of motion is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \text{... (3.34)}$$

Here,

$$\frac{\partial f}{\partial y} = 0 \quad and \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x(1+y'^2)}} \qquad \text{... (3.35)}$$

Using relation (3.35) in (3.34), we have

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\therefore\ \frac{d}{dx}\left(\frac{y'}{\sqrt{x(1+y'^2)}}\right) = 0$$

$$\therefore\ \frac{y'^2}{x(1+y'^2)} = \frac{1}{2a} \quad where\ 2a\ is\ const.$$

$$\therefore 2ay'^2 = x + xy'^2$$

$$\therefore\ y'^2(2a - x) = x$$

$$\therefore\ y' = \frac{\sqrt{x}}{\sqrt{2a - x}}$$

$$\therefore y = \int \frac{\sqrt{x}}{\sqrt{2a - x}}\, dx \qquad \text{.... (3.36)}$$

To solve above equation, we substitute

$$x = a(1 - \cos\theta) \qquad \text{... (3.37)}$$

$$\therefore dx = a\ \sin\theta\ d\theta \quad and \quad \sqrt{\frac{x}{2a - x}} = \tan\theta/2$$

∴ we get,

$$y = \int a(1 - \cos\theta)\ d\theta$$

$$\therefore y = a(\theta - \sin\theta) + const \quad ...(3.38)$$

At origin, when $x = y = 0$, $\theta = 0$ then, $const = 0$

Thus, we have the equations as

$$\left.\begin{aligned} x &= a(1-\cos\theta) \\ y &= a(\theta - \sin\theta) \end{aligned}\right\} \quad ...(3.39)$$

These are the equations of a cycloid passing through the origin. Thus, the path of the particle is a cycloid as shown in Fig.(3.3)

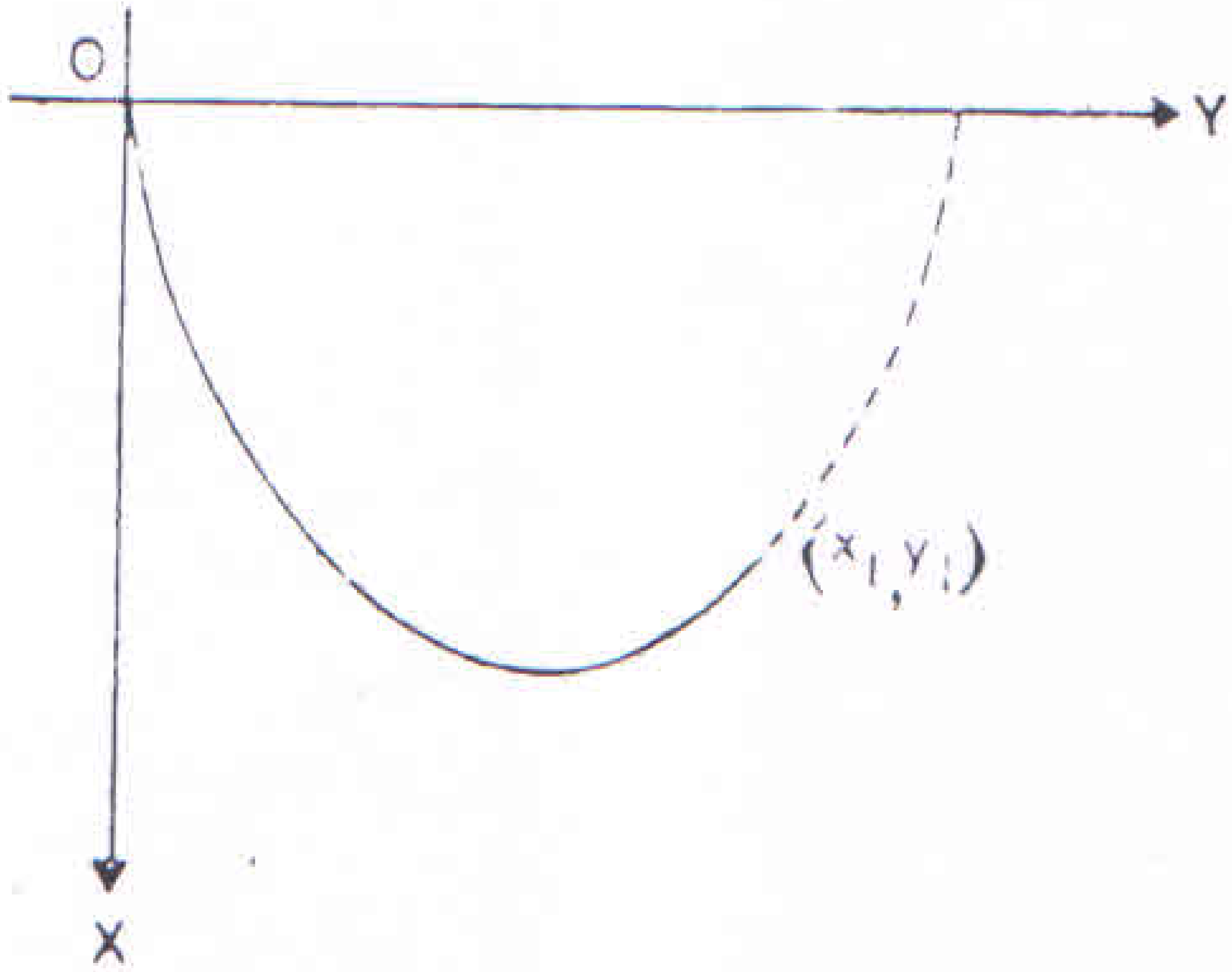

Fig: 3.3

The value of constant a must be adjusted such that the particle passes through the other points (x_1, y_1). Along this path, the time of transit of the particle from the origin to (x_1, y_1) is found to be a minimum.

3. **Show that the geodesics of a spherical surface are great circles, i.e. the circles whose centers lie at the centre of the sphere.**

A geodesic is a line which represents the shortest path between any two points, when the path is restricted to be on some surface. The surface is a spherical surface.

The element of distance ds on the surface of a sphere of radius r, in the spherical coordinates is given by

$$ds^2 = r^2[d\theta^2 + sin^2\theta\, d\emptyset^2]$$

$$\therefore ds = r[d\theta^2 + sin^2\theta\, d\emptyset^2]^{1/2} \quad ...(3.40)$$

The total distance between two points having coordinates $(r_1, \theta_1, \emptyset_1)$ and $(r_2, \theta_2, \emptyset_2)$ is given by

$$J = \int_1^2 ds = \int_{\theta_1}^{\theta_2} r \left[1 + sin^2\theta \left(\frac{d\emptyset}{d\theta}\right)^2\right]^{1/2} d\theta \quad ...(3.41)$$

Here, functional f is given by

$$f = r\left[1 + sin^2\theta \left(\frac{d\emptyset}{d\theta}\right)^2\right]^{1/2} \quad ...(3.42)$$

If J is to be extremum, we must have

$$\frac{\partial f}{\partial \emptyset} - \frac{d}{d\theta}\left(\frac{\partial f}{\partial \emptyset'}\right) = 0 \qquad ...(3.43)$$

Where, $\emptyset' = \frac{d\emptyset}{d\theta}$, Since, $\frac{\partial f}{\partial \emptyset}$

The Euler-Lagrange's equation becomes

$$\frac{d}{d\theta}\left(\frac{\partial f}{\partial \emptyset'}\right) = 0$$

$$\therefore \frac{d}{d\theta}\left[\frac{\partial}{\partial \emptyset'}\left\{r\left(1 + sin^2\theta\, \emptyset'^2\right)^{1/2}\right\}\right] = 0$$

$$\therefore \frac{d}{d\theta}\left[\frac{1}{2}\left(1 + sin^2\theta\, \emptyset'^2\right)^{1/2} sin^2\theta\, 2\emptyset'\right] = 0$$

$$\therefore \frac{d}{d\theta}\left[\frac{sin^2\theta\, \emptyset'}{\left(1 + sin^2\theta\, \emptyset'^2\right)^{1/2}}\right] = 0$$

Since $r \neq 0$ and is constant.

Integration of above equation gives,

$$\frac{sin^2\theta\, \emptyset'}{\left(1 + sin^2\theta\, \emptyset'^2\right)^{1/2}} = c \qquad ...(3.44)$$

Where c is constant. Above equation becomes

$$sin^4\theta\ \ \emptyset'^2 = c^2\left(1 + sin^2\theta\, \emptyset'^2\right)$$

$$\therefore \emptyset'^2 sin^2\theta(sin^2\theta - c^2) = c^2$$

$$\emptyset' = \frac{c}{sin\theta\ (sin^2\theta - c^2)^{1/2}} = \frac{c\ cosec^2\theta}{(1 - c^2 - c^2cot^2\theta)^{1/2}} \qquad ...(3.45)$$

Using standard integral, we get

$$\emptyset = \alpha - sin^{-1}(k\ cot\theta) \qquad ...(3.46)$$

Where, α and $k = \frac{c}{\sqrt{1-c^2}}$ are contants.

This gives

$$k\ cot\theta = sin(\alpha - \emptyset)$$

Or

$$k\ cos\theta = sin(\alpha - \emptyset)\ sin\theta \qquad ...(3.47)$$

Using relation between the Cartesian coordinates (x, y, z) and the spherical polar coordinates $(r, \theta, \emptyset)$, we can write above relation as

$$zk = x\ sin\alpha - y\ cos\alpha \qquad ...(3.48)$$

$where,\ \ x^2 + y^2 + z^2 = r^2$

Equation (3.48) represents a plane passing through the origin and hence cutting the surface of the sphere in a great circle. Thus, the extremum value of the distance between the two points on the surface of a sphere is an arc of a circle whose centre lies at the centre of the sphere.

Hamilton's Principle:

Statement: *All the possible paths along which a dynamical system may move from one point to another within a given interval of time, the actual path followed is that which minimizes the time interval of the Lagrangian.*

This principle can also be stated as:

The motion of the system from instant t_1 to instant t_2 is such that the line integral

$$J = \int_{t_1}^{t_2} L \; dt \qquad ...(3.49)$$

Where, $L = T - V$, is an extremum for the path of the motion.

In terms of the calculus of variation, we can state Hamilton's principle as,

$$\delta J = \delta \int_{t_1}^{t_2} L \; dt = 0 \qquad ...(3.50)$$

with variation zero at $t = t_1$ and $t = t_2$.The line integral of L is an extremum.

Lagrangian L can be expressed as,

$$L = L(q_1, q_2, \dots \dots \dots q_n, \dot{q}_1, \dot{q}_2, \dots \dots \dots \dot{q}_n, t) \qquad ...(3.51)$$

Therefore, equation (3.50) becomes

$$\delta J = \delta \int_{t_1}^{t_2} L(q_1, q_2, \dots \dots \dots q_n, \dot{q}_1, \dot{q}_2, \dots \dots \dots \dot{q}_n, t) \; dt = 0$$

$$\therefore \; \delta J = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) \; dt = 0 \qquad ...(3.52)$$

The Euler's equation of motion is

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \qquad where, i = 1,2, \dots . n$$

Using above equation the Lagrange's equation of motion becomes

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$$

$$\therefore \; \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad ...(3.53)$$

This is Lagrange's equation of motion.

In terms of Lagrangian L , Hamilton's principle can be stated as:

All the possible path, along which dynamical system may move from one point to another in the configuration space within a given interval of time, the actual path followed is that for which the time interval of the Lagrangian function for the system is an extremum.

Equivalence of Lagrange's and Newton's Equations:

We shall show that Newton's formulation is equivalent to the Lagrangian formulation by obtaining Newton's law of motion from Lagrange's equation and Hamilton's principle from Newton's equation.

1. Newton's equation of motion from Lagrange's equation:

The Lagrangian equation of motion for a single particle in rectangular coordinates x_i is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0, where \; i = 1,2,3 \qquad ...(3.54)$$

But, $L = T - V$

$$\therefore \frac{d}{dt}\left[\frac{\partial(T-V)}{\partial \dot{x}_i}\right]-\frac{\partial(T-V)}{\partial x_i}=0 \qquad ...(3.55)$$

For conservative system, kinetic energy is a function of velocity $\dot{x}_i$, and potential energy is a function of potential x_i only.

$$\therefore \frac{\partial T}{\partial x_i}=0 \quad and \ \frac{\partial V}{\partial \dot{x}_i}=0 \qquad ...(3.56)$$

Hence, equation (3.56) becomes,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)+\frac{\partial V}{\partial x_i}=0$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)=-\frac{\partial V}{\partial x_i} \qquad ...(3.57)$$

For a conservative system, we have

$$-\frac{\partial V}{\partial x_i}=F_i \qquad ...(3.58)$$

Also,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)=\frac{d}{dt}\frac{\partial}{\partial \dot{x}_i}\left(\sum_{i=1}^{3}\frac{1}{2}m\dot{x}_i^2\right)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)=\frac{d}{dt}(m\dot{x}_i)=\frac{d}{dt}(p_i)$$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right)=\dot{p}_i \qquad ...(3.59)$$

Using equations (3.58) & (3.59) in (3.54), we get

$$F_i=\dot{p}_i \qquad ...(3.60)$$

This is Newton's equation of motion. Hence, Lagrangian and Newtonian equations are equivalent.

2. Hamilton's Principle from Newton's Equation:

Consider a case of a single particle. Let $x_i(t), i=1,2,3$ or $\vec{r}(t)$ be the solution of Newton's equation.

Let $\vec{r}_1(t_1)$ and $\vec{r}_2(t_2)$ be the position vectors representing the position of the particle at instants t_1 and t_2.

Then, Newton's equations and the equations of the constraints are at every point along the path of the particle.

Consider another path having the same end points and travelled in the same interval of the time (t_2-t_1). Then such a path would be represented by

$$\vec{r}(t)\rightarrow\vec{r}(t)+\delta\vec{r}(t) \qquad ...(3.61)$$

Since the end points are the same for both the paths.

$$\therefore \delta\vec{r}(t_1)=\delta\vec{r}(t_2)=0 \qquad ...(3.62)$$

Newton's equation of motion is

$$\vec{F}=m\vec{a}=m\ddot{\vec{r}} \qquad ...(3.63)$$

Let δW be the work done in passing from the true path to the varied path.

$$\therefore \delta W=\vec{F}\cdot\delta\vec{r}$$

$$\therefore \delta W = m\ddot{\vec{r}} \cdot \delta\vec{r} \qquad \text{... (3.64)}$$

The total force $\vec{F}$acting on the particle is the vector sum of the applied force $\vec{F}_a$ and the force of constraints $\vec{F}_c$.

$$\therefore \vec{F} = \vec{F}_a + \vec{F}_c \qquad \text{... (3.65)}$$

The varied path considered above is such that no work is done by the force of constraint.

$$\therefore \vec{F}_c \cdot \delta\vec{r} = 0 \qquad \text{... (3.66)}$$

Therefore, equation (3.64) becomes

$$\delta W = \vec{F}_a \cdot \delta\vec{r} \qquad \text{... (3.67)}$$

If the applied force $\vec{F}_a$ is a conservative force and hence is derivable from a potential energy function V, then

$$\vec{F}_a \cdot \delta\vec{r} = -\delta V \qquad \text{... (3.68)}$$

Hence, equation (3.64) becomes

$$-\delta V = m\ddot{\vec{r}} \cdot \delta\vec{r} \qquad \text{... (3.69)}$$

Now consider,

$$\frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) = \dot{\vec{r}} \cdot \frac{d}{dt}(\delta\vec{r}) + \ddot{\vec{r}} \cdot \delta\vec{r} \qquad \text{... (3.70)}$$

$$\therefore \frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) = \dot{\vec{r}} \cdot \delta\dot{\vec{r}} + \ddot{\vec{r}} \cdot \delta\vec{r}$$

$$\therefore \frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) = \vec{v} \cdot \delta(\vec{v}) + \ddot{\vec{r}} \cdot \delta\vec{r}$$

$$\therefore \frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) = \delta\left(\frac{1}{2}v^2\right) + \ddot{\vec{r}} \cdot \delta\vec{r}$$

$$\therefore \ddot{\vec{r}} \cdot \delta\vec{r} = \frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) - \delta\left(\frac{1}{2}v^2\right) \qquad \text{... (3.71)}$$

Multiplying throughout by m, we get

$$m\ddot{\vec{r}} \cdot \delta\vec{r} = m\frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) - \delta\left(\frac{1}{2}mv^2\right)$$

Using equation (3.69) in above relation, we have

$$-\delta V = m\frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) - \delta T$$

$$\therefore \delta T - \delta V = m\frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) \qquad \text{... (3.72)}$$

Now integrating between the time t_1 and t_2,

$$\int_{t_1}^{t_2} \delta(T - V)\ dt = m\int_{t_1}^{t_2} \frac{d}{dt}\left(\dot{\vec{r}} \cdot \delta\vec{r}\right) dt = m\int_{t_1}^{t_2} d\left(\dot{\vec{r}} \cdot \delta\vec{r}\right)$$

$$\therefore \int_{t_1}^{t_2} \delta(T - V)\ dt = m\left[\dot{\vec{r}} \cdot \delta\vec{r}\right]_{t_1}^{t_2}$$

But, $\delta\vec{r} = 0$ at the end points, we get

$$\int_{t_1}^{t_2} \delta(T - V)\ dt = 0$$

$$\therefore \delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

$$\therefore \delta \int_{t_1}^{t_2} L\, dt = 0 \qquad \text{... (3.73)}$$

Which is Hamilton's principal.

Advantages of the Lagrangian Formulation – Elecromechanical Analogies:

1. Hamilton's principle contains all the mechanics of the holonomic conservative system. The principle involves to set up the Lagrangian by finding the kinetic energy and potential energy using suitable generalised coordinates. The Lagrangian formulation is invariant with respect to the choice of the coordinate system.
2. The Lagrangian formulation is not restricted to the mechanical system only. It includes non-mechanical systems such as elastic field, electromagnetic field, acoustical systems etc.

➢ **Illustration:**

1. L-C-R series connection:

Let us consider an electrical circuit containing an inductance L, resistance R, and capacitance C connected in series as shown in fig.(3.4)

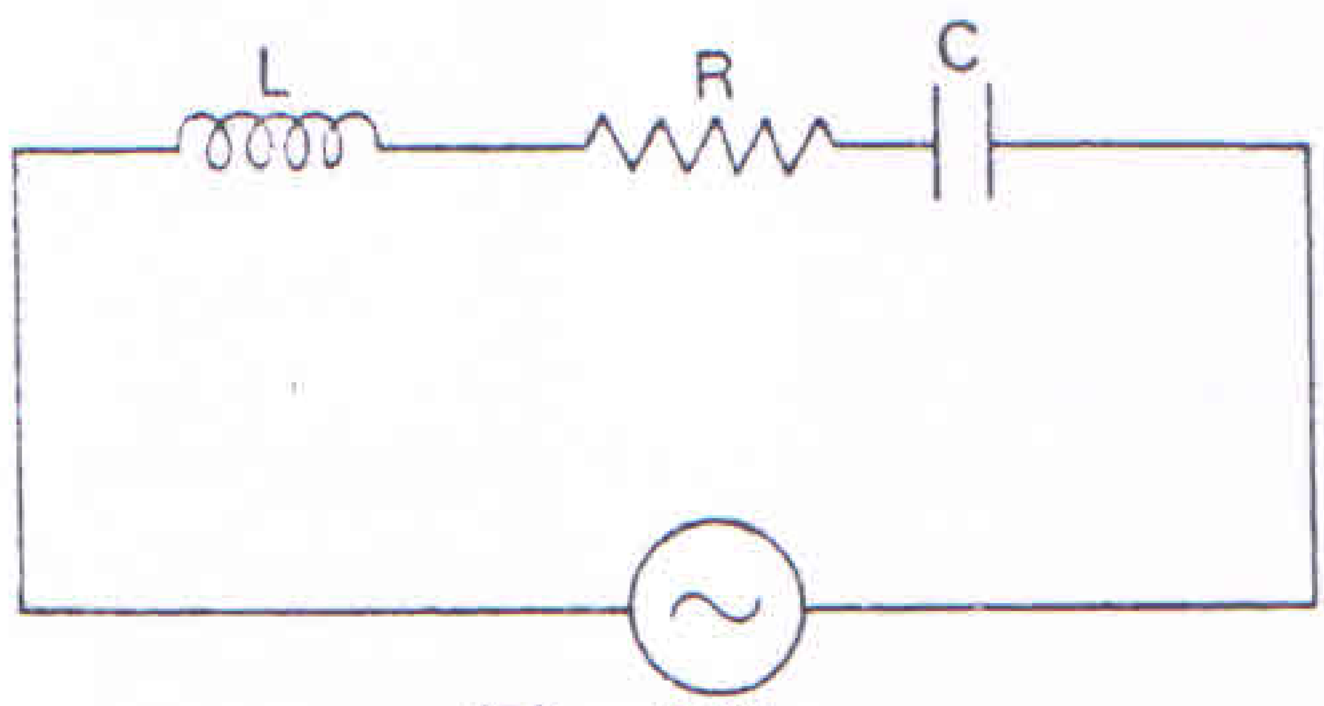

Fig.3.4

The voltage U across each element is

$$\left.\begin{aligned} U &= L\frac{dI}{dt} \\ U &= R\, I \\ U &= \frac{1}{C}\int I\, dt \end{aligned}\right\} \qquad \text{... (3.74)}$$

The external electromotive force $\mathcal{E}(t)$ is the sum of voltage across each elements of the circuit. Thus we have,

$$L\frac{dI}{dt} + R\, I + \frac{1}{C}\int I\, dt = \mathcal{E}(t) \qquad \text{... (3.75)}$$

But, $I = \frac{dq}{dt}$, where q is the charge. Hence, above equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}(t) \qquad ...(3.76)$$

This equation is similar to the equation for the forced oscillator which is given by

$$m\frac{d^2x}{dt^2} + 2m\mu\frac{dx}{dt} + kx = F(t) \qquad ...(3.77)$$

From above equations (3.76) & (3.77), we haveInductance Lcorresponds to inertial mass m , Ohmic resistance$R$ correspond to dissipative constants μ and capacitance $\frac{1}{C}$ to force constant k. The charge $q$plays the role of coordinate x and e.m.f. $\mathcal{E}(t)$ of the external force $F(t)$.

Now comparing equations (3.76) and (3.77), we get

$$L = R, \qquad R = 2m\mu, \quad \frac{1}{C} = k, \quad q = x \quad and \qquad \mathcal{E}(t) = F(t) \qquad ...(3.78)$$

With these comparisons, we can write

$$\left.\begin{aligned} Kinetic\ energy\ T &= \frac{1}{2}mv^2 = \frac{1}{2}L\dot{q}^2 \\ Potential\ energy\ V &= \frac{1}{2}kx^2 = \frac{q^2}{2C} \\ Dissipation\ function\ \mathcal{F} &= m\mu\dot{x}^2 = \frac{1}{2}R\dot{q}^2 \\ Generalised\ force\ Q(t) &= \mathcal{E}(t) \end{aligned}\right\} \qquad ....(3.79)$$

The Lagrangian $L = T - V$

Hence for series connection, the Lagrangian is given by

$$L = \frac{1}{2}L\dot{q}^2 - \frac{q^2}{2C} \qquad ...(3.80)$$

2. L-C-R parallel connection:

Now, consider a circuit contains L, R and C in parallel with e.m.f. $\mathcal{E}(t)$ as shown in Fig.(3.5)

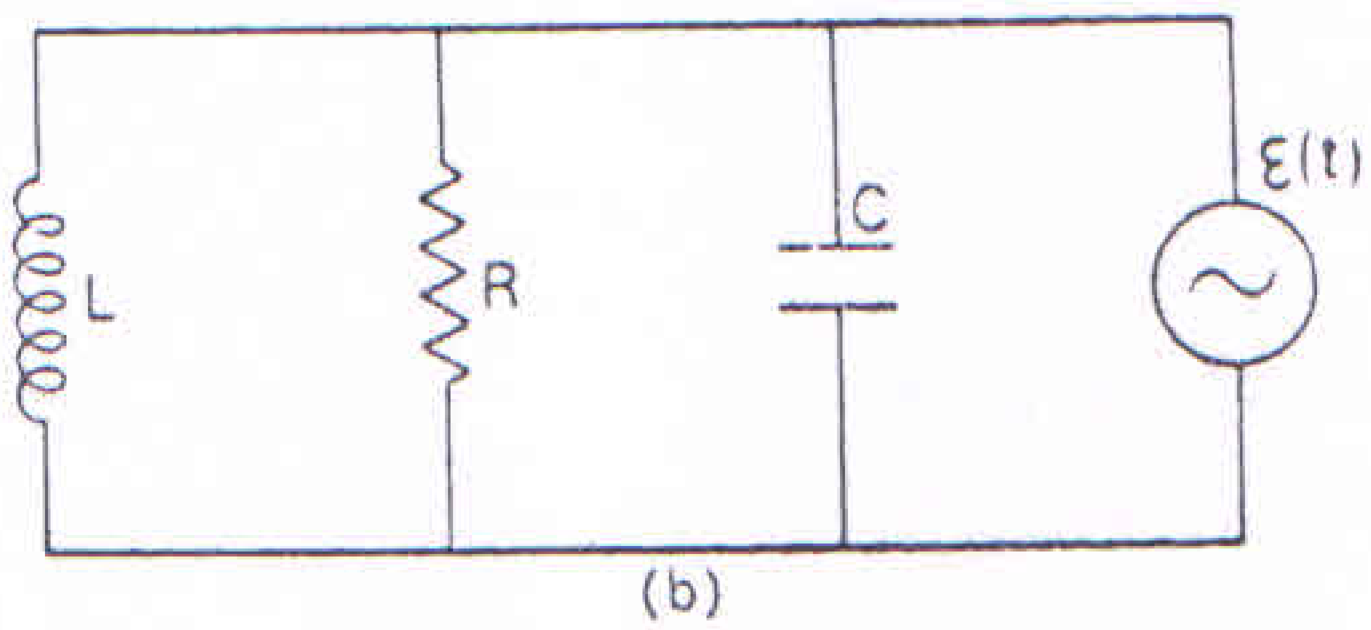

Fig:3.5

In this case, the potential difference across each element is the same, but the current flowing through R, L and C add to give the total current.

The current through R is $\frac{U}{R}$, through L is $\frac{1}{L}\int U\,dt$ and through C is $C\frac{dU}{dt}$.

$$\therefore\ \frac{U}{R} + \frac{1}{L}\int U\,dt + C\frac{dU}{dt} = I(t) \qquad ....(3.81)$$

Differentiate above equation with respect to time, we get

$$C\frac{d^2U}{dt^2}+\frac{1}{R}\frac{dU}{dt}+\frac{U}{L}=\frac{dI}{dt} \qquad ...(3.82)$$

This equation is similar to the equation for the forced oscillator which is given by

$$m\frac{d^2x}{dt^2}+2m\mu\frac{dx}{dt}+kx=F(t) \qquad ...(3.83)$$

Now comparing equations (3.82) and (3.83), we get

$$x=U, \quad m=C, \ 2m\mu=\frac{1}{R}, \ k=\frac{1}{L} \quad and \quad F(t)=\frac{dI}{dt} \qquad ...(3.84)$$

With these comparisons, we can write

$$\left.\begin{aligned} Kinetic\ energy\ T &= \frac{1}{2}mv^2=\frac{1}{2}C\dot{U}^2 \\ Potential\ energy\ V &= \frac{1}{2}kx^2=\frac{U^2}{2L} \\ Dissipation\ function\ \mathcal{F} &= m\mu\dot{x}^2=\frac{1}{2R}\dot{U}^2 \\ Generalised\ force\ Q(t) &= \frac{dI}{dt} \end{aligned}\right\} \qquad ...(3.85)$$

The Lagrangian $L=T-V$

Hence for parallel connection, the Lagrangian is given by

$$L=\frac{1}{2}C\dot{U}^2-\frac{U^2}{2L} \qquad ...(3.86)$$

Lagrange's Undetermined Multipliers:

What is necessity of undetermined multipliers?

In the constrained motion of the physical system, the degrees of freedom are reduced. We use the equation of constraint to eliminate the dependent variables and set a new independent variables. Sometimes it is difficult or inconvenient to eliminate the dependent variables. Under these circumstances, use of Lagrange's multipliers gives an alternative technique to solve the problem.

Consider a function $f=f(x,y,z)$ of three independent variables. The function f has an extremum value when

$$df=0 \qquad ...(3.87)$$

$$\therefore df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz \qquad ...(3.88)$$

The necessary and sufficient condition to satisfied equation (3.87) is

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=\frac{\partial f}{\partial z}=0 \qquad ...(3.89)$$

Let the equation of constraint be

$$g(x,y,z)=0 \qquad ...(3.90)$$

$$\therefore dg=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy+\frac{\partial g}{\partial z}dz \qquad ...(3.88)$$

Because of the equation of constraint (3.90), now independent variables are two, say x and y.

Now multiplying equation (3.88) by λ and add in equation (3.88), we get

$$df+\lambda\,dg=\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}\right)dz=0 \quad ...(3.89)$$

The multiplier λ can be chosen by setting

$$\frac{\partial f}{\partial z}+\lambda\frac{\partial g}{\partial z}=0 \quad ...(3.90)$$

Where, we assume that $\frac{\partial g}{\partial z}\neq 0$.

Using equation (3.90) in (3.89), we have

$$\left(\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}\right)dy=0 \quad ...(3.91)$$

Since x and y are independent, their coefficients must vanish separately.

Hence,

$$\left.\begin{aligned}\frac{\partial f}{\partial x}+\lambda\frac{\partial g}{\partial x}=0\\ \frac{\partial f}{\partial y}+\lambda\frac{\partial g}{\partial y}=0\end{aligned}\right\} \quad ...(3.92)$$

Thus, when equation (3.90) & (3.92) are satisfied, we get $df=0$ or f has an extremum value.

We have now four variables x, y, z, and λ , and three equations (3.90) & (3.92). The forth equation is actually the equation of constraint.

In the solution, we want to know only x, y and z. The multiplier λ need not to be determined. For this reason, it is called Lagrange's undetermined multiplier.

ILLUSTRATION:

Consider a quantum mechanical problem of a particle of mass m in a box. It is in the form of rectangular parallelepiped of sides x, y and z.

The ground state energy of a particle in the box is given by

$$E=\frac{h^2}{8m}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) \quad ...(3.93)$$

We want to find out the shape of the box which will give minimum energy with the condition that the volume must be constant.

$$\therefore V(x,y,z)=xyz=c \quad ...(3.94)$$

Here $f=E$ and the equation of constraint is

$$g(x,y,z)=xyz-c=0 \quad ...(3.95)$$

Using equations (3.90) & (3.92), we have

$$\frac{\partial E}{\partial x}+\lambda\frac{\partial g}{\partial x}=-\frac{h^2}{4mx^3}+\lambda yz=0 \quad ...(3.96)$$

$$\frac{\partial E}{\partial y}+\lambda\frac{\partial g}{\partial y}=-\frac{h^2}{4my^3}+\lambda xz=0 \quad ...(3.97)$$

$$\frac{\partial E}{\partial z}+\lambda\frac{\partial g}{\partial z}=-\frac{h^2}{4mz^3}+\lambda xy=0 \quad ...(3.98)$$

Multiplying equation (3.96) by x, (3.97) by y and (3.98) by z , we get

$$-\frac{h^2}{4mx^2}+\lambda xyz=0$$

$$\therefore\ \lambda xyz=\frac{h^2}{4mx^2} \qquad \text{... (3.99)}$$

Similarly,

$$\lambda xyz=\frac{h^2}{4my^2} \qquad \text{... (3.100)}$$

$$\lambda xyz=\frac{h^2}{4mz^2} \qquad \text{... (3.101)}$$

From above equations, we have

$$\lambda xyz=\frac{h^2}{4mx^2}=\frac{h^2}{4my^2}=\frac{h^2}{4mz^2} \qquad \text{... (3.102)}$$

Hence the condition for minimum energy is

$$x=y=z \qquad \text{... (3.103)}$$

Hence box should be a cube. Here, the multiplier λ is undetermined.

Lagrange's Equations for Non-Holonomic Systems:

Let us consider the equation of constraints

$$\sum_k a_{lk}dq_k+a_{lt}dt=0 \qquad \text{... (3.104)}$$

$$Here, l=1,2,\dots\dots m$$

Above equation represents the m relations of constraints between the differential's of $'q'$.

Since the Hamilton's principle does not involve variation in time, the virtual displacement must satisfy the equation

$$\sum_k a_{lk}\delta q_k=0 \qquad \text{... (3.105)}$$

$$Here, l=1,2,\dots\dots m$$

Above equation can be used to reduce the number of virtual displacement. For this we use method of Lagrange's undetermined multipliers

$$\therefore\ \lambda_l\sum_k a_{lk}\delta q_k=0 \qquad \text{... (3.106)}$$

The m equations expressed as equation (3.106) are now combined with

$$\int_{t_1}^{t_2}\left[\sum_k\left(\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}_k}\right)\delta q_k\right]dt=0 \qquad \text{.... (3.107)}$$

for the conservative system.

For this, we sum up equation (3.106) over l and then integrate from t_1 to t_2.

$$\therefore\ \int_{t_1}^{t_2}\sum_{kl}\lambda_l\,a_{lk}\delta q_k\,dt=0 \qquad \text{... (3.108)}$$

Adding equations (3.107) & (3.108)

$$\int_{t_1}^{t_2}\left[\sum_{k=1}^{n}\left(\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}+\sum_{l}\lambda_l\, a_{lk}\right)\delta q_k\right]dt=0 \qquad \dots(3.109)$$

We may choose λ such that

$$\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}+\sum_{l}\lambda_l\, a_{lk}=0 \qquad \dots(3.110)$$

$$where, \quad k=(n-m-1),(n-m),\dots\dots\dots n$$

If this is true, we must write equation (3.109) as,

$$\int_{t_1}^{t_2}\left[\sum_{k=1}^{n-m}\left(\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}+\sum_{l}\lambda_l\, a_{lk}\right)\right]dt=0 \qquad \dots(3.111)$$

Above equation involve those q_k that are independent.

Above equation satisfied only if

$$\frac{\partial L}{\partial q_k}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}+\sum_{l}\lambda_l\, a_{lk}=0 \qquad \dots(3.112)$$

$$where, \quad k=1,2,\dots\dots\dots..(n-m)$$

Combining equations (3.110) & (3.112), we get complete set of Lagrange's equations for non-holonomic systems.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=\sum_{l}\lambda_l\, a_{lk} \qquad \dots(3.113)$$

$$where, \quad k=1,2,\dots\dots\dots..n$$

Now, equation (3.104) can be written as

$$\sum_{k}a_{lk}\dot{q}_k+a_{lt}=0 \qquad \dots(3.114)$$

$$where, \quad l=1,2,\dots\dots\dots..m$$

To understand the physical significance of 'λ' ,consider a system on which an external force Q_k' acts instead of the constraints. Hence we can write

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k}-\frac{\partial L}{\partial q_k}=Q_k' \qquad \dots(3.115)$$

These equations must be identical with equation (3.113).

Hence,

$$\sum_{l}\lambda_l\, a_{lk}=\ Q_k' \qquad \dots(3.116)$$

Thus, $\sum_l \lambda_l\, a_{lk}$ can be treated as generalised force of constraints.

Now, an equation of the non-holonomic constraint is

$$f(q_1,q_2,\dots\dots..q_n,t)=0$$

$$\therefore\ \sum_{k}\frac{\partial f}{\partial q_k}dq_k+\frac{\partial f}{\partial t}dt=0 \qquad \dots(3.117)$$

Comparing equation (3.117) with equation (3.104), we get

$$a_{lk}=\frac{\partial f}{\partial q_k} \qquad and \qquad a_{lt}=\frac{\partial f}{\partial t} \qquad \dots(3.118)$$

Hence, Lagrangian method of undetermined multiplier can be used for holonomic constraints if

(1) It is inconvenient to reduce all the coordinates of the system to independent

(2) The force of constraints are required

Applications of the Lagrangian method of undetermined multipliers:

(a) Cylinder Rolling on Inclined Plane:

Consider a cylinder rolling without slipping on an inclined plane as shown in Fig.(3.6). We wish to find the acceleration of the cylinder and the frictional force of constraint.

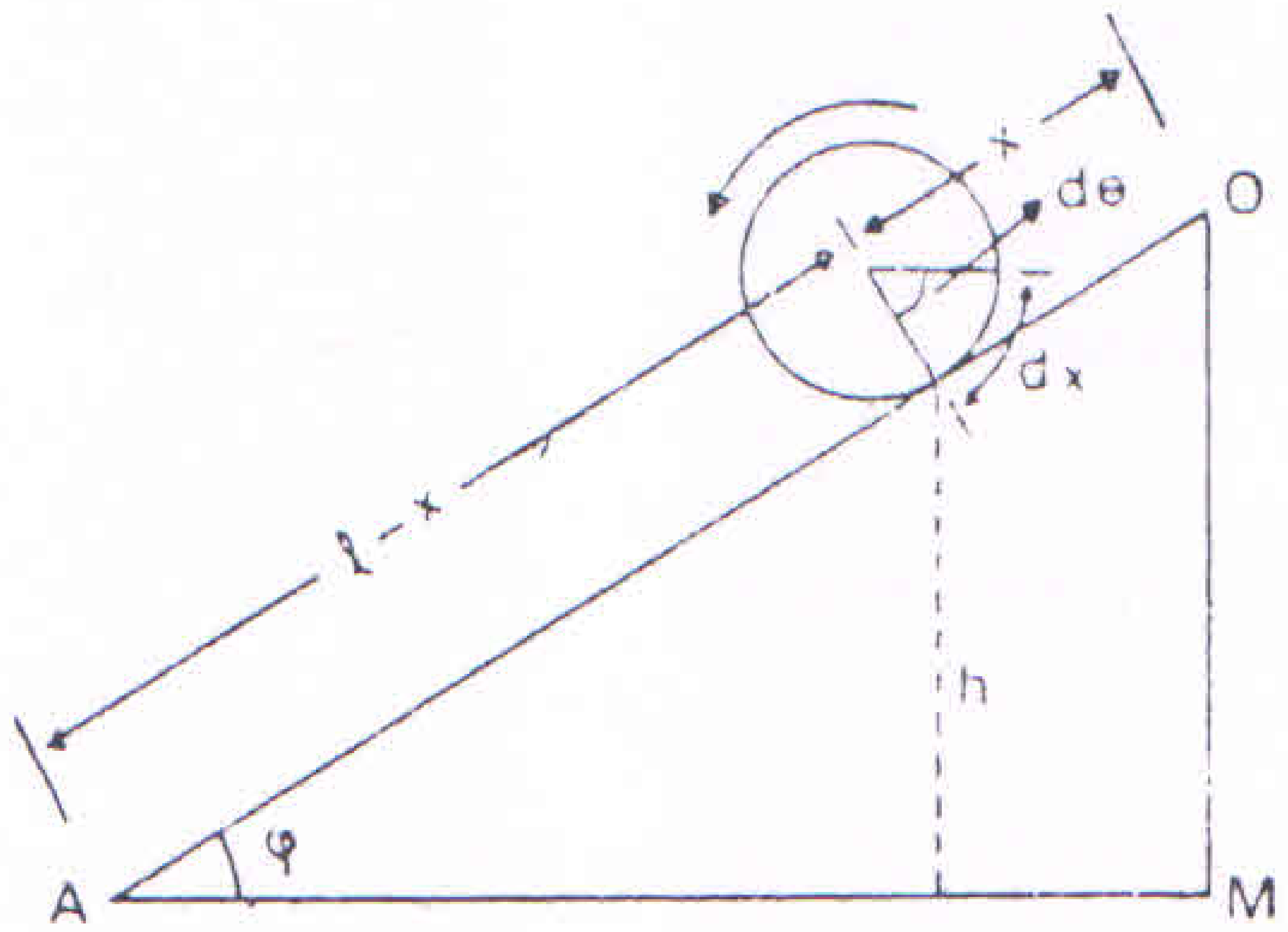

Fig.:3.6

Let ϕ be the angle of inclination of the given plane of length l with the horizontal. Let the cylinder of radius r start from the point O and roll down the plane along the line of greatest slope without slipping. Then, the equation of constraint is

$$r\,d\theta = dx$$

$$\therefore\ r\,d\theta - dx = 0 \qquad ...(3.119)$$

We know the equation of constraint as,

$$\sum_k a_{lk} dq_k + a_{lt} dt = 0 \qquad ...(3.120)$$

In this case, there are two variables θ and x. Hence, we can write

$$a_\theta d\theta + a_x dx = 0 \qquad ....(3.121)$$

Now, compare equations (3.119) & (3.121), we have

$$a_\theta = r \qquad and \qquad a_x = -1 \qquad ....(3.122)$$

The kinetic energy of cylinder is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$$

But, the moment of inertia of the cylinder is

$$I = \frac{1}{2}mr^2$$

Hence,

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}mr^2\dot{\theta}^2 \qquad \dots(3.123)$$

The potential energy is

$$V = mgh = mg(l - x)sin\phi \qquad \dots.(3.124)$$

Hence, the Lagrangian is given by

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}mr^2\dot{\theta}^2 - mg(l - x)sin\phi \qquad \dots.(3.125)$$

The Lagrangian equations in terms of x and θ are,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = a_x\lambda \quad and \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = a_\theta\lambda$$

$$\therefore \frac{d}{dt}(m\dot{x}) - [-mg(-1)sin\phi] = -\lambda \quad and \quad \frac{d}{dt}\left(\frac{1}{2}mr^2\dot{\theta}\right) - 0 = r\,\lambda$$

$$\therefore m\ddot{x} - mgsin\phi + \lambda = 0 \qquad \dots(3.126)$$

And,

$$\frac{1}{2}mr^2\ddot{\theta} - r\,\lambda = 0 \qquad \dots.(3.127)$$

Now, equation (3.119) becomes

$$r\,\ddot{\theta} - \ddot{x} = 0$$

$$\therefore r\,\ddot{\theta} = \ddot{x} \qquad \dots(3.128)$$

Using equation (3.128) in (3.127), we have

$$\frac{1}{2}mr\,\ddot{\theta} = \lambda$$

$$\therefore \frac{1}{2}m\ddot{x} = \lambda$$

$$\therefore \lambda = \frac{m\ddot{x}}{2} \qquad \dots(3.129)$$

Substituting this value of λ in equation (3.126), we get

$$m\ddot{x} - mgsin\phi + \frac{m\ddot{x}}{2} = 0$$

$$\therefore \frac{3}{2}m\ddot{x} = mgsin\phi$$

$$\therefore \ddot{x} = \frac{2gsin\phi}{3} \qquad \dots(3.130)$$

Using equation (3.128) in (3.130), we get

$$r\,\ddot{\theta} = \frac{2gsin\phi}{3}$$

$$\therefore \ddot{\theta} = \frac{2gsin\phi}{3r} \qquad \dots(3.131)$$

The frictional force of constraint λ is given by

$$\lambda = \frac{m\ddot{x}}{2} = \frac{m}{2}\frac{2gsin\phi}{3} = \frac{mgsin\phi}{3} \qquad \dots(3.132)$$

We can write,

$$\ddot{x} = v\frac{dv}{dx} = \frac{2gsin\phi}{3}$$

Integrating above equation, we get

$$v = \sqrt{\frac{4glsin\phi}{3}} \qquad ... (3.133)$$

This is the velocity at the bottom of the inclined plane.

(b) Simple Pendulum:

Let (r, θ)be the coordinates of the bob of the pendulum with respect to the point O of the support as shown in Fig.(3.7).

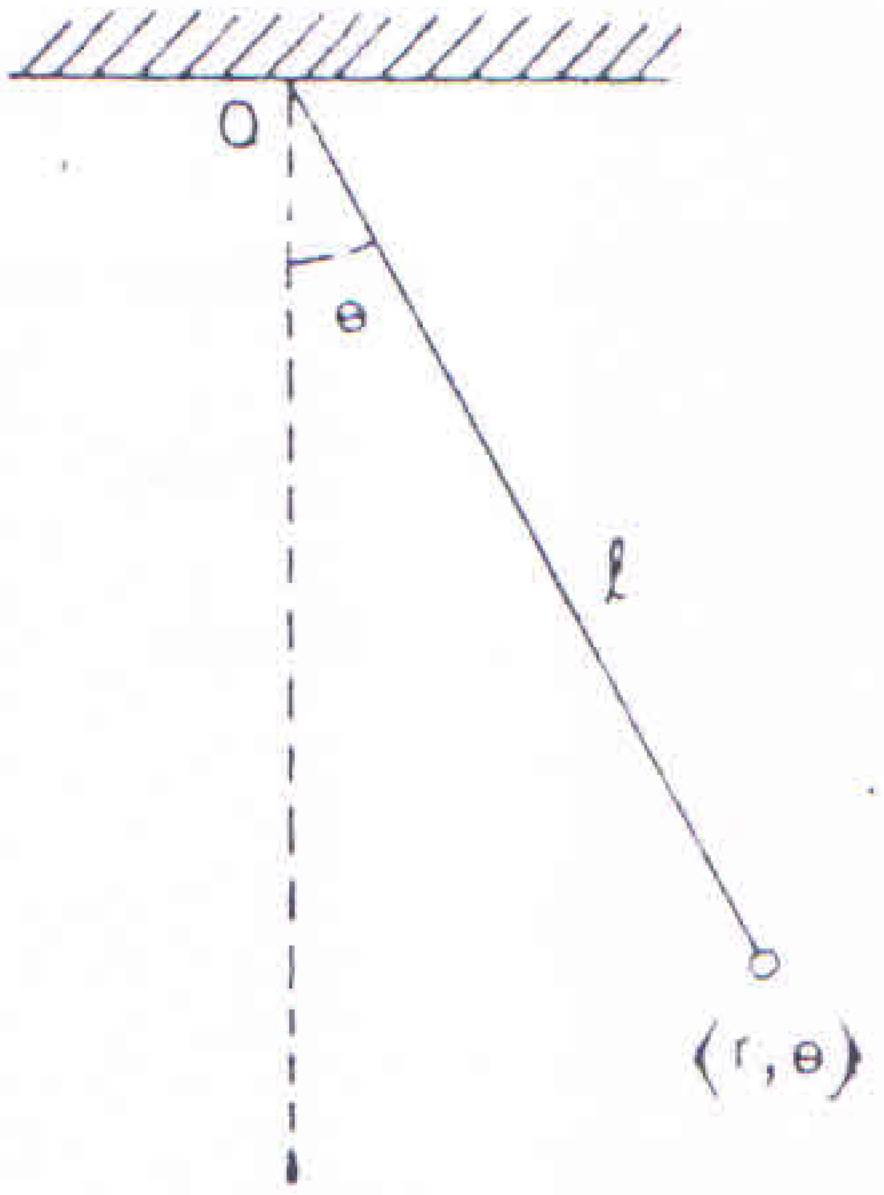

Fig: 3.7

The Lagrangian of simple pendulum is given by

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mgr\ \cos\theta \qquad ... (3.134)$$

The equation of constraint is

$$r - l = 0$$

$$\therefore dr = 0 \qquad ... (3.135)$$

We know the equation of constraint as,

$$\sum_k a_{lk}dq_k + a_{lt}dt = 0 \qquad ... (3.136)$$

In this case, we can write

$$a_r dr + a_\theta d\theta = 0 \qquad .... (3.137)$$

Now, compare equations (3.135) & (3.137), we have

$$a_r = 1 \qquad and \qquad a_\theta = 0 \qquad .... (3.138)$$

Lagrange's equations in terms of r and θ are expressed as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = a_r\lambda$$

$$\therefore \frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 - mg\cos\theta = \lambda$$

$$\therefore m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda \qquad ... (3.139)$$

Similarly,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=a_\theta \lambda$$

$$\therefore \frac{d}{dt}\left(mr^2\dot{\theta}\right)+mgr\ \sin\theta=0$$

$$\therefore mr^2\ddot{\theta}+mgr\ \sin\theta=0 \qquad \text{... (3.140)}$$

This is required equation of motion of simple pendulum.

(c) Particle on Sphere:

Let us consider a particle of mass m moving under the action of gravity on the surface of a smooth sphere of radius l.

We want to find its equation of motion and the angle θ_c at which the particle flies off from the surface.

Let the origin of the coordinates be the centre of the sphere and let the z- axis be vertically upwards.

The equation of constraint is given by

$$r-l=0 \qquad \text{... (3.141)}$$

Where, r is the radial distance of the particle and $l=const.$

$$dr=0 \qquad \text{... (3.142)}$$

We know the equation of constraint as,

$$\sum_k a_{lk}dq_k+a_{lt}dt=0 \qquad \text{... (3.143)}$$

In this case, we can write

$$a_r dr+a_\theta d\theta+a_\phi d\phi=0 \qquad \text{.... (3.144)}$$

Now, compare equations (3.142) & (3.144), we have

$$a_r=1\ ,\quad a_\theta=0\quad and\quad a_\phi=0 \qquad \text{.... (3.145)}$$

Let us suppose that the particle is initially at rest and it slide down along the surface. For convenient, we take $\phi=0$.

The Lagrangian for the particle is

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+mgr\ \cos\theta \qquad \text{... (3.146)}$$

The Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=a_r\lambda$$

$$\therefore \frac{d}{dt}(m\dot{r})-mr\dot{\theta}^2+mg\cos\theta=\lambda$$

$$\therefore m\ddot{r}-mr\dot{\theta}^2+mg\cos\theta=\lambda$$

But, $r=l$ Hence, $\dot{r}=\ddot{r}=0$

$$\therefore -ml\dot{\theta}^2+mg\cos\theta=\lambda \qquad \text{... (3.147)}$$

Also,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)-\frac{\partial L}{\partial \theta}=a_\theta \lambda$$

$$\therefore \frac{d}{dt}\left(mr^2\dot{\theta}\right)-mgl\ \sin\theta=0$$

$$\therefore\ mr^2\ddot{\theta} - mgl\ \sin\theta = 0$$

But, $r = l$

$$\therefore\ ml^2\ddot{\theta} - mgl\ \sin\theta = 0 \qquad \dots(3.148)$$

The undetermined multiplier λ is dependent on θ.

Now, differentiate equation (3.147) with respect to $'\,t\,'$, we get

$$-2ml\dot{\theta}\ddot{\theta} - mg\sin\theta\,\dot{\theta} = \frac{d\lambda}{d\theta}\dot{\theta}$$

$$\therefore\ -2ml\ddot{\theta} - mg\sin\theta = \frac{d\lambda}{d\theta}$$

Now, substituting value of $\ddot{\theta}$ from equation (3.148) in above expression, we have

$$-2ml\left(\frac{mgl\sin\theta}{ml^2}\right) - mg\sin\theta = \frac{d\lambda}{d\theta}$$

$$\therefore\ -2mg\sin\theta - mg\sin\theta = \frac{d\lambda}{d\theta}$$

$$\therefore\ -3mg\sin\theta = \frac{d\lambda}{d\theta} \qquad \dots(3.149)$$

Integrating above equation, we get

$$\lambda(\theta) = 3mg\cos\theta + c \qquad \dots(3.150)$$

From equation (3.147), when $\theta = 0$

$$\lambda = mg \qquad \dots(3.151)$$

This is the force of constraint at the top of the sphere.

Substituting this value of λ in equation (3.150) for $\theta = 0$, we get

$$c = -2mg$$

Putting this value of c in equation (3.150), we have

$$\lambda(\theta) = 3mg\cos\theta - 2mg \qquad \dots(3.152)$$

The particle will move on the surface as long as the force of constraint is positive. The condition is

$$\lambda(\theta) = 3mg\cos\theta - 2mg \ \geq 0 \qquad \dots(3.152)$$

This equality is true if

$$\cos\theta_c = \frac{2}{3} \qquad \dots(3.153)$$

i.e. at the angle $\theta_c = \cos^{-1}\frac{2}{3}$, the particle flies off the surface. Here, we neglected the friction of the surface.

(d) The Schrodinger Wave Equation:

The Lagrangian formulation is an important in modern physics. Consider a quantum mechanical problem of variation in

$$\delta\int \Psi^*(r)H(r,p)\Psi(r)d\tau = 0 \qquad \dots(3.154)$$

The constraint is that total probability is conserved.

Hence,

$$\int \Psi^*(r)\Psi(r)d\tau = 0 \qquad \dots(3.155)$$

Equation (3.154) states that the energy of a particle described by the wave function Ψ is an extremum with the condition given by equation (3.155) that the total probability of finding the particle in the whole space is unity.

Here, H is the quantum mechanical Hamiltonian operator for a particle of mass m

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(r) \qquad \text{... (3.156)}$$

Here, $V(r)$ is the potential field in which the particle is moving.

The wave function of the particle Ψ and its complex conjugate are treated as independent variables.

Substituting equation (3.156) in (3.154) and solving by integration by parts, we have

$$\delta \int \left[\frac{\hbar^2}{2m}\vec{\nabla}\Psi^* \cdot \vec{\nabla}\Psi + V\Psi^*\Psi\right] d\tau = 0 \qquad \text{... (3.157)}$$

Combining equation (3.157) and (3.155) with undetermined multiplier λ, we get

$$\delta \int \left[\frac{\hbar^2}{2m}\vec{\nabla}\Psi^* \cdot \vec{\nabla}\Psi + V\Psi^*\Psi - \lambda\Psi^*\Psi\right] d\tau = 0 \qquad \text{... (3.158)}$$

$$\therefore f = \frac{\hbar^2}{2m}\vec{\nabla}\Psi^* \cdot \vec{\nabla}\Psi + V\Psi^*\Psi - \lambda\Psi^*\Psi \qquad \text{... (3.159)}$$

Where f is a function of Ψ and Ψ^*.

Using the Euler-Lagrange equation, we get

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V(r)\Psi = \lambda\Psi \qquad \text{... (3.160)}$$

Here, λ is a real constant and represents the energy of the physical quantum mechanical system.

Thus, the Lagrangian formulation through variational methods is not simply an hypothetical concept but provide a powerful tool in the study of physical phenomena.

Question Bank

Multiple choice questions:

(1) The n-dimensional space is called ____________ space

(a) solar (b) **configuration**

(c)real (d) zero

(2) In variational principle the line integral of some function between two end points is ____________

(a) zero (b) infinite

(c) **extremum** (d) one

(3) The shortest distance between two points in a plane is ___________

(a) circular (b) hyperbolic

(c)parabolic (d) **straight line**

(4) The path of a particle is _________when it is moving under constant conservative force field

(a) **cycloid** (b) hyperbolic

(c)parabolic (d) straight line

(5) The equation of constraints is _______ for a cylinder rolling on inclined plane

(a) $r\ d\emptyset - dx = 0$ (b) $\boldsymbol{r\ d\theta - dx = 0}$

(c) $r\ dr - dx = 0$ (d) $r\ dx - dx = 0$

(6) The equation of constraints for a simple pendulum is _________

$(a)\ r\ d\theta - l = 0$ (b) $r + l = 0$

(c) $r\ d\theta + l = 0$ (d) $\boldsymbol{r\ - l = 0}$

(7) The angle of flies off for a particle moving on spherical surface is __________

(a) $\phi_C = cos^{-1}\left(\frac{3}{2}\right)$ (b) $\phi_C = sin^{-1}\left(\frac{3}{2}\right)$

(c) $\boldsymbol{\phi_C = cos^{-1}\left(\frac{2}{3}\right)}$ (d) $\phi_C = sin^{-1}\left(\frac{3}{2}\right)$

Short Questions:

1. What is configuration space?
2. State the variational principle
3. Define geodesic line
4. Write the equation of cycloid when a particle is moving in a constant conservative force field
5. State the Hamilton's principle
6. Show that the Lagrangian and Newtonian equation are equivalent
7. What is undetermined multiplier?
8. Write the Lagrangian for a cylinder rolling on inclined plane
9. Write the Lagrangian of simple pendulum in terms of spherical polar coordinates
10. Write the Hamilton's equation of motion

Long Questions:

1. Describe the configuration space
2. Discuss the technique of calculus of variation and derive the general Euler's equation
3. Derive the Euler's equation using δ- notation
4. To show that the shortest distance between two points in a plane is a straight line
5. Discuss the shortest time problem for a motion of a particle in a constant conservative force field
6. Show that the extremum value of the distance between the two points on the surface of a sphere is an arc of a circle whose centre lies at the centre of the sphere
7. State the Hamilton's principle and derive the Lagrange's equation of motion

8. Derive the Hamilton's principle from Newtonian formulation
9. Construct the Lagrangian for series and parallel connection of inductance L, resistance R and capacitor C with an external electromotive force $\varepsilon(t)$
10. Describe the Lagrange's undetermined multiplier with illustration
11. Derive the Lagrange's equation of motion for Non-holonomic system
12. Construct the Lagrangian and derive the equations of motion for a cylinder rolling on inclined plane using undetermined multiplier
13. Derive the equation of motion for a simple pendulum using undetermined multiplier
14. Construct the Lagrangian for motion of a particle on a sphere and derive the equations of motion using undetermined multiplier
15. Derive the Schrodinger wave equation using variational principle
16. Derive the Hamilton's equation of motion